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Prolongation structures for Harry Dym type equations and Bäcklund transformations of CC ideals

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Abstract. We examine the prolongation structures for a certain class of differential equations. The interrelation between the incomplete set of commutator relations and the defining relations for Kac–Moody algebras is discussed. Constant coefficient ideals are introduced for which Bäcklund transformations are found in a purely algebraic manner.

1. Introduction

Applying the Wahlquist–Estabrook prolongation technique [1, 2] to nonlinear partial differential equations in two dimensions often leads to infinite-dimensional Kac–Moody algebras as prolongation algebras. In most cases one has to deduce the structure of the Lie algebra by introducing new generators for unknown commutators and going through the Jacobi identities. Repeating that procedure (by computer) until the commutator table is large enough it is then a question of intuition and experience whether one can identify the algebra.

Nevertheless, we should mention that the investigation of the incomplete set of commutator relations with regard to automorphisms which are due to Lie point symmetries of the corresponding first-order system of differential equations (isogroup) is helpful for discovering the algebra.

In sections 2 and 3 we shall determine the prolongation structure of a certain class of differential equations and show in the infinite-dimensional cases that the incomplete set of commutator relations is nothing but the defining relations for Kac–Moody algebras. Thus no further work on unveiling the structure of the algebra will be necessary.

Using these prolongation structures we shall discusss in section 4 the corresponding constant coefficient (CC) ideals which admit transformations between apparently different equations by interchanging coordinate and potential.

The subject of section 5 will be the formulation of Bäcklund transformations in a purely algebraic manner. It will be seen that it has been advantageous for this purpose to introduce the concept of CC ideals.

2. The class of differential equations

We are considering equations of the form

$$\frac{\partial}{\partial t}u = u^n \left(\frac{\partial}{\partial x}\right)^{k+1} u \qquad k \geqslant 1 \tag{1}$$

which we call 'Harry Dym type equations' because a prominent member is the Harry Dym equation for $k = 2, n = 3$. Another well-known equation of this class is the case $k = 3, n = 2$; it arises in general relativity in the context of algebraically special radiative spacetimes, the so-called Robinson–Trautman class, as a specialization to axisymmetric solutions. We shall not be concerned with the linear equation $n = 0$.

The two-forms pertinent to equation (1) can be written as

$$\begin{aligned} \omega_i &= \mathrm{d}u_i\,\mathrm{d}t - u_{i+1}\,\mathrm{d}x\,\mathrm{d}t \qquad & i &= 0,\dots,k-1 \\ \omega_k &= \mathrm{d}u_0\,\mathrm{d}x + u^n\,\mathrm{d}u_k\,\mathrm{d}t \qquad & u_0 &= u. \end{aligned}$$

The ωs pulled back to the solution manifold (x, t) and annulled give our original equation; $\omega_i = 0$ are nothing but the definition of the derivatives of u while $\omega_k = 0$ yields the equation. It is easy to see that the set of two-forms ω is a closed differential ideal, i.e. $\mathrm{d}\omega = 0 \bmod \omega$.

We now look for pseudopotentials y^α and introduce one-forms Ω^α such that

$$\Omega^\alpha = -\mathrm{d}y^\alpha + F^\alpha(u_i, y^\beta)\,\mathrm{d}x + G^\alpha(u_i, y^\beta)\,\mathrm{d}t \tag{2}$$

and

$$\mathrm{d}\Omega^\alpha = 0 \bmod (\omega, \Omega^\beta). \tag{3}$$

From now on we suppress Greek indices on F and G and write G_i for the derivative of G with respect to u_i. In addition, we formally set $G_r := 0$ for $r < 0$ or $r > k$.

Collecting terms in $\mathrm{d}x\,\mathrm{d}t$ respectively $\mathrm{d}u_i\,\mathrm{d}x$ we get

$$\begin{aligned} [F, G] &= \sum_{i=0}^{k-1} u_{i+1} G_i \\ G_k &= u_0^n F_0 \\ F_i &= 0 \qquad i \geqslant 1. \end{aligned} \tag{4}$$

The commutator between F and G is taken with respect to the ys only and is (up to the sign) the usual Lie bracket between vector fields. We now have to determine the dependence of F and G on the u_i. First we note that

$$\begin{aligned} G_{ik} &= 0 \qquad i \geqslant 1 \\ [F, G_k] &= u_1 G_{0k} + G_{k-1} \end{aligned}$$

and consequently

$$\begin{aligned} G_{0k} + G_{1,k-1} &= 0 \\ G_{i,k-1} &= 0 \qquad i \geqslant 2. \end{aligned} \tag{5}$$

Thus we get, in general,

$$G_{k-l,l} + G_{k-l-1,l+1} = 0 \qquad l \leqslant k/2$$
$$G_{k-l,i} = 0 \qquad i > l.$$

Taking the commutator of this equation with F we derive—note $[F_0, G_k] = 0$—

$$G_{k-l,l-1} + 2G_{k-l-1,l} + G_{k-l-2,l+1} = 0.$$

This is a linear algebraic system for $G_{k-l,l-1}$, $l \leqslant k/2$, whose determinant does not vanish. Consequently

$$G_{k-l,l-1} = 0 \qquad l \leqslant k/2. \tag{6}$$

In particular this implies $G_{k-1,0} = 0$ and hence $G_{k00} = 0$. Thus we find already

$$F = \alpha X + \beta B_k + Y$$
$$\alpha, \beta = \begin{cases} u_0 & \ln u_0 & \text{for } n = 1 \\ \ln u_0 & -u_0^{-1} & \text{for } n = 2 \\ (2-n)^{-1}u_0^{2-n} & (1-n)^{-1}u_0^{1-n} & \text{otherwise} \end{cases} \tag{7}$$

where X, B_k and Y are vector fields depending only on the pseudopotentials. Now we have

$$[F, G_{k-l}] = \sum_{i=0}^{l-2} u_{i+1} G_{k-l,i} + G_{k-l-1} + (-1)^l u_{l+1} X \tag{8}$$

and thus with (6)

$$G_{k-l,l-2} + G_{k-l-1,l-1} = 0 \qquad 2 \leqslant l \leqslant k/2.$$

This is again a linear algebraic system of equations giving $G_{k-l,l-2}$ in terms of $G_{k-2,0}$. From (8) it follows that

$$2[F, G_{k-2,0}] + 2[F_0, G_{k-2}] - [F, G_{k-3,1}] = u_1(2G_{k-2,00} - G_{k-3,10})$$

and thus $G_{k-2,00} = 0$. From the explicit form of F and G_k we can obtain G_{k-1} and G_{k-2} through (8) and show that $G_{k-2,00} = 0$ implies $G_{k-2,0} = 0$. Thus

$$G_{k-l,l-2} = 0.$$

Repeating these arguments one can now show that G has to have the following form:

$$G = \tfrac{1}{2}\sum_{i=0}^{k}(-1)^i u_i u_{k-i} X + \sum_{i=0}^{k} u_i B_i + f(u_0) Z_0 + C. \tag{9}$$

X, B_i, Z_0 and C depend only on the ys. For odd k X vanishes identically as can be seen from (5). Inserting F and G now into the commutator (4) and collecting terms in u_i gives

$$\tfrac{1}{2}\sum_{i=0}^{k}(-1)^i u_i u_{k-i} \qquad [Y, X] = 0 \tag{10}$$

$$\tfrac{1}{2}\beta\sum_{i=0}^{k}(-1)^i u_i u_{k-i} \qquad [B_k, X] = 0 \tag{11}$$

$$l \geqslant 2 : \alpha u_l \qquad [X, B_l] = 0 \tag{12}$$

$$l \geqslant 2 : \beta u_l \qquad [B_k, B_l] = 0 \tag{13}$$

$$l \geqslant 2 : u_l \qquad [Y, B_l] = B_{l-1} \tag{14}$$

$$u_1 : \alpha[X, B_1] + \beta[B_k, B_1] + [Y, B_1] = B_0 + f_0 Z_0 \tag{15}$$

$$\begin{aligned} u_0 : \alpha f[X, Z_0] + \alpha[X, C] + \beta u_0[B_k, B_0] + \beta f[B_k, Z_0] \\ + \beta[B_k, C] + u_0[Y, B_0] + f[Y, Z_0] + [Y, C] = 0. \end{aligned} \tag{16}$$

From the Jacobi identity for $[X,[Y,B_2]]$ follows immediately that $[X, B_1] = 0$ and thus $f_0 = \beta$ or

$$f = \begin{cases} u_0 \ln u_0 - u_0 & \text{for } n = 1 \\ -\ln u_0 & \text{for } n = 2 \\ \dfrac{1}{(2-n)(1-n)} u_0^{2-n} & \text{otherwise} \end{cases}$$

and

$$[B_k, B_1] = Z_0 \qquad [Y, B_1] = B_0. \tag{17}$$

The equation containing u_0 terms gives

$$\begin{aligned} n = 1 :& [B_k, Z_0] = [X, Z_0] = [B_k, C] = [Y, C] = 0 \\ & [B_k, B_0] + [Y, Z_0] = [X, C] + [Y, B_0] - [Y, Z_0] = 0 \end{aligned} \tag{18}$$

$$\begin{aligned} n = 2 :& [B_k, Z_0] = [X, Z_0] = [B_k, C] = [Y, B_0] = 0 \\ & [B_k, B_0] - [Y, C] = [X, C] - [Y, Z_0] = 0 \end{aligned} \tag{19}$$

$$\begin{aligned} n \neq 1, 2 :& [X, Z_0] = [B_k, C] = [Y, B_0] = 0 \\ & (2-n)[B_k, B_0] + [Y, Z_0] + (1-n)[X, C] = 0 \\ & [B_k, Z_0] + 8[Y, C] = 0 \qquad \text{for } n = \tfrac{3}{2} \\ & [B_k, Z_0] = [Y, C] = 0 \qquad \text{otherwise.} \end{aligned} \tag{20}$$

Before we resolve the special cases let us see what information can be extracted from (10)–(16). First we consider (13) and (14). It follows in general that

$$[B_i, B_l] = 0 \qquad i + l \geqslant k + 2.$$

(17) implies

$$Z_0 = (-1)^l [B_{k-l}, B_{l+1}]. \tag{21}$$

This gives immediately

$$k \text{ odd} \Rightarrow Z_0 = 0. \tag{22}$$

Moreover

$$Z_1 := [Y, Z_0] = (-1)^l ([B_{k-l}, B_l] + [B_{k-l-1}, B_{l+1}]).$$

Summing this expression we get

$$(p - q + 1) Z_1 = (-1)^q [B_{k-q}, B_q] + (-1)^p [B_{k-p-1}, B_{p+1}]$$

and thus for $p = (k/2 - 1)$ (taking k as even)

$$(k/2 - l) Z_1 = (-1)^l [B_{k-l}, B_l]. \tag{23}$$

It is now easy to dispose of the cases $k =$ odd, $n = 1$ and $n = 2$. As has been noted already, X and Z_0 vanish identically for odd k. Hence we have

$$\begin{aligned} k \text{ odd}: n = 2 &: [B_{k-l}, B_{l+1}] = (-1)^l Z_0 \qquad [Y, B_l] = B_{l-1} \\ n \neq 2 &: Z_0 = 0 \\ &[.,.] = 0 \qquad \text{otherwise}. \end{aligned}$$

There is the exceptional case $k = 1$, $n = 2$. Using z defined by $\mathrm{d}z = (-1/u)\,\mathrm{d}x + u_1\,\mathrm{d}t$ instead of x as coordinate one shows that the equation is in fact equivalent to the linear equation $u_t = u_{zz}$ and thus of no further interest to us.

From now on k will be taken as even. For $n = 1$ it follows from (18) and (23) that $Z_1 = 0$. Hence

$$\begin{aligned} &[B_{k-l}, B_{l+1}] = (-1)^l Z_0 \qquad [Y, B_l] = B_{l-1} \\ &[X, C] = -[Y, B_0] = D \qquad [.,.] = 0 \qquad \text{otherwise}. \end{aligned}$$

Again using (23) we get for $n = 2$

$$\begin{aligned} &[B_{k-l}, B_{l+1}] = (-1)^l Z_0 \qquad [B_{k-l}, B_l] = (-1)^l (k/2 - l) Z_1 \\ &[Y, B_l] = B_{l-1} \qquad [Y, Z_1] = [X, C] = [Y, C] = Z_1 \\ &[.,.] = 0 \qquad \text{otherwise} \\ &k \neq 2 \Rightarrow Z_1 = 0. \end{aligned}$$

Having found that the special cases for n give a finite dimensional prolongation algebra we shall consider $n \neq 1, 2$. For $k > 2$ (21) implies

$$[B_k, Z_0] = 0$$

and hence the special case $n = \frac{3}{2}$ is relevant only for $k = 2$. However, this equation can be linearized again by using z—$\mathrm{d}z = -(1/u)\,\mathrm{d}x + u_2\,\mathrm{d}t$—as coordinate.

In the general case it follows from (23) and (20) that

$$\frac{k}{2} Z_1 = [B_k, B_0] = \frac{1}{n-2}(Z_1 + (1-n)[X, C])$$

and thus

$$\left(\frac{k}{2}(n-2) - 1\right) Z_1 = (1-n)[X, C].$$

It can easily be shown that $[X, C]$ commutes with all other generators and, as can be seen from the arguments given later, the algebra will close. Hence the interesting case is

$$n = 2 + \frac{2}{k}. \tag{24}$$

In what follows we shall assume n to have this particular value. Commuting Y with (23) gives

$$[Y, Z_1] := Z_2$$
$$(k/2 - l) Z_2 = (-1)^l ([B_{k-l-1}, B_k] + [B_{k-l}, B_{k-1}]).$$

Summing this relation from zero to some value of l and noting that all but one term on the right-hand-side cancel we get

$$(l+1)\frac{k-l}{2} Z_2 = (-1)^l [B_{k-l-1}, B_l]. \tag{25}$$

Now we repeat this procedure to get the commutator of Y and Z_2. We find, however,

$$\frac{1}{2}(l+1)(l+2)\left(\frac{k}{2} - \frac{l+3}{3}\right)[Y, Z_2] = (-1)^l [B_{k-l-2}, B_l]$$

and thus if we let $l = (k/2) - 1$, that Y and Z_2 have to commute. Therefore we conclude

$$[B_i, B_l] = 0 \qquad k - 2 < i + 1 < k + 2. \tag{26}$$

The next commutators to be determined are those between the Bs and Zs. We find from (21)

$$[B_i, Z_0] \sim [B_{k-l}, [B_i, B_{l+1}]] - [B_{l+1}, [B_i, B_{k-l}]].$$

The relation (26) gives

$$[B_i, Z_0] \neq 0 \qquad k-3 < i < \frac{k}{2}+2, \frac{k}{2}-3 < i < 2 \tag{27}$$

and similarly

$$[B_i, Z_1] \neq 0 \qquad k-2 < i < \frac{k}{2}+3, \frac{k}{2}-3 < i < 2 \tag{28}$$

$$[B_i, Z_2] \neq 0 \qquad k-2 < i < \frac{k}{2}+3, \frac{k}{2}-2 < i < 3. \tag{29}$$

For $k > 6$ these relations imply that all such commutators vanish. For $k = 6$ we are left with $i = 1,4$; $i = 1,5$; $i = 2,4$ respectively as non-vanishing commutators for the three Zs. Using the relation

$$[Y, [B_i, Z_l]] = [B_{i-1}, Z_l] + [B_i, Z_{l+1}]$$

they can also be shown to vanish and thus all Z commute with all B. We have determined all commutators for $k \geqslant 6$ or $n \neq 2 + (2/k)$; to list them again:

$$\begin{aligned}
&[Y, B_i] = B_{i-1} \\
&[B_{k-l}, B_{l+1}] = (-1)^l Z_0 \\
&[B_{k-l}, B_l] = (-1)^l (k/2 - l) Z_1 \\
&[B_{k-l-1}, B_l] = (-1)^l (l+1)(k - l/2) Z_2 \\
&[.,.] = 0 \qquad \text{otherwise} \\
&n \neq 2 + (2/k): \qquad \left(\frac{k}{2}(n-2) - 1\right) Z_1 = (1-n)[X, C] \qquad Z_2 = 0.
\end{aligned} \tag{30}$$

To summarize: The only equations of the Harry Dym type which admit an infinite-dimensional prolongation algebra are, apart from the linear equations $n = 0$ and those which can be linearized, i.e. $k = 1, n = 2; k = 2, n = \frac{3}{2}$, the Harry Dym equation itself and the equation $u_t = u^{5/2} u_{xxxxx}$ [3].

3. The algebra

In section 2 we saw that for k odd or $k \geqslant 6$ or $n \neq 2+(2/k)$ either the corresponding differential equation is equivalent to a linear equation or the algebra closes to a finite-dimensional nilpotent Lie algebra. Even though the widespread conjecture that a differential equation which admits only a finite-dimensional prolongation algebra does not possess a Bäcklund transformation is not true [4] we shall be dealing in the following only with the remaining cases $k = 2, n = 3$ and $k = 4, n = (5/2)$.

If we regard (14) and (17) as defining relations for the generators B_i, $i = 0, \ldots, k-1$ all other commutator relations can be summarized in the following way:

(1) The generators X and C commute with all other generators.

(2) The rest of the algebra is freely generated on the generators B_k and Y with constraints

(3) $(\mathrm{ad}\,B_k)^3Y = 0 \qquad (\mathrm{ad}\,Y)^3B_k = 0 \qquad$ for $k = 2$

 $(\mathrm{ad}\,B_k)^2Y = 0 \qquad (\mathrm{ad}\,Y)^5B_k = 0 \qquad$ for $k = 4$.

Now we recall the defining relations for *Kac–Moody algebras* [5] generated by $3n$ generators e_i, f_i and h_i:

$$[h_i, h_j] = 0 \qquad [e_i, f_j] = \delta_{i,j} h_i$$
$$[h_i, e_j] = a_{ij} e_j \qquad [h_i, f_j] = -a_{ij} f_j$$

and the *Serré relations*

$$(\mathrm{ad}\,e_i)^{1-a_{ij}} e_j = 0 \qquad (\mathrm{ad}\,f_i)^{1-a_{ij}} f_j = 0 \tag{31}$$

where $A = (a_{ij})_{i,j=0,\ldots,n-1}$ is the *generalized Cartan matrix.*

The only two-dimensional Cartan matrices with rank 1 correspond to the untwisted affine algebra

$$A_1^{(1)} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$

and the twisted affine algebra ($\tau = 2$)

$$A_2^{(2)} = \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}.$$

Thus if we identify B_k with e_0 and Y with e_1 for $k = 2$, $k = 4$ we find that the prolongation algebra is a subalgebra of the Kac–Moody algebra $A_1^{(1)}$ and $A_2^{(2)}$ respectively (we forget about the Abelian generators X and C).

Using an isomorphism of $A_1^{(1)}$ (respectively $A_2^{(2)}$) (without centre and derivation) to a graded subalgebra of the loop algebra of the simple Lie algebra $\mathrm{sl}(2,\mathbb{R})$ (respectively $\mathrm{sl}(3,\mathbb{R})$) we find for $k = 2$:

$$Y = X_3^1 \qquad Z_0 = -X_2^3 \qquad B_0 = -X_3^3$$
$$B_1 = X_1^2 \qquad B_2 = -X_2^1 \tag{32}$$

and for $k = 4$:

$$Y = X_5^0 \qquad Z_0 = X_4^2 \qquad B_0 = X_7^1 \qquad B_1 = X_6^1$$
$$B_2 = X_2^1 \qquad B_3 = X_3^1 \qquad B_4 = -\tfrac{1}{2}X_8^1 \tag{33}$$

where $X_i^j = X_i \otimes \lambda^j$ is an element of the loop algebra $\mathrm{sl}(r,\mathbb{R}) \otimes \mathbb{R}(\lambda, \lambda^{-1})$ and $X_1, \ldots, X_{r^2-1}$, $r = 2$ (respectively $r = 3$), is a certain basis of $\mathrm{sl}(2,\mathbb{R})$ (respectively $\mathrm{sl}(3,\mathbb{R})$). A linear representation of that particular basis from which one can extract the structure constants will be given in section 5.

4. The CC ideal

In section 5 we shall see that aside from other advantages it is convenient for getting Bäcklund transformations to write a differential equation (and therefore a whole class of differential equations!) as a *constant coefficient ideal* (CC ideal) or *invariant differential system* originally expounded by Harrison [6] and Estabrook [7] respectively. Let us briefly recall what a CC ideal is.

We are considering an (in general) infinite-dimensional Lie algebra

$$[X_i, X_j] = c^k{}_{ij} X_k \qquad c^k{}_{ij} = \text{constant}. \tag{34}$$

The generators X_i are vector fields on an (in general) infinite-dimensional manifold with coordinates y^α (pseudopotentials). We have one-forms dual to the vectors and define two-forms by

$$\omega^k := \xi^k - \tfrac{1}{2} c^k{}_{ij} \xi^i \xi^j .$$

The two-forms ω form a closed differential ideal, i.e. $\mathrm{d}\omega = 0 \bmod \omega$, because the structure constants $c^k{}_{ij}$ satisfy the Jacobi identities. This guarantees that the equations $\omega = 0$ are integrable (Cartan's calculus of exterior differential forms).

The vector-valued one-form

$$\Omega := -\mathrm{d}y + X_i \xi^i$$

again has the property $\mathrm{d}\Omega = 0 \bmod (\Omega, \omega)$, i.e. the equations $\Omega = 0$ are also integrable.

For the purpose of obtaining differential equations from the system of two-forms, we have to set almost all one-forms ξ^i to zero:

$$\xi^i = 0 \qquad i \notin J, J \subset \mathbb{Z}.$$

Then the ω system splits into two sets:

$$\begin{aligned} \rho^i &:= \mathrm{d}\xi^i - \tfrac{1}{2} c^i{}_{jk} \xi^j \xi^k \qquad && \text{for } i \in J \\ \sigma^i &:= \qquad\quad c^i{}_{jk} \xi^j \xi^k && \text{for } i \notin J. \end{aligned}$$

The integrability conditions $\mathrm{d}(\rho, \sigma) = 0 \bmod (\rho, \sigma)$ are still satisfied.

The maximum dimension of an integral manifold of the (ρ, σ) system is Cartan's *genus* g. The one-forms ξ^i can thus be written as linear combinations of g linearly independent one-forms $\eta^j, j = 1, \ldots, g$:

$$\xi^i = a^i_j \eta^j \qquad i \in J$$

with functions a^i_j depending on the coordinates of the integral manifold.

The σ forms have to vanish identically, whereas $\rho = 0$ gives first-order differential equations to be combined to one (or more) higher order equation.

The linearly independent one-forms can be found by looking for exact one-forms within the ρ system to be used as coordinate differentials on the integral manifold. In case we find more than g exact one-forms, only g of them will be linearly independent and it is our choice which of them to use; the others will be differentials of potentials. This allows transformations between apparently different equations by *interchanging coordinate and potential.*

4.1. The affine algebra $A_1^{(1)}$

The non-vanishing one-forms dual to generators of $A_1^{(1)}$ in which we are interested (cf (32)) are

$$\begin{array}{ll} \xi^1 \text{ dual to } X_2^1 & \xi^2 \text{ dual to } X_3^1 \\ \xi^3 \text{ dual to } X_1^2 & \xi^4 \text{ dual to } X_2^3 \\ \xi^5 \text{ dual to } X_3^3 & \end{array}$$

which define the corresponding CC ideal (on $\rho = \sigma = 0$)

$$\begin{array}{ll} \mathrm{d}\xi^1 = 0 & \xi^1\xi^5 + \xi^4\xi^2 = 0 \\ \mathrm{d}\xi^2 = 0 & \xi^3\xi^4 = 0 \\ \mathrm{d}\xi^3 = \xi^1\xi^2 & \xi^3\xi^5 = 0 \\ \mathrm{d}\xi^4 = \xi^1\xi^3 & \xi^4\xi^5 = 0 \\ \xi^5 = \xi^3\xi^2. & \end{array} \tag{35}$$

Without going into the detailed calculations (cf, e.g., [8] on how this formalism works), we state that after solving the algebraic σ system of (35) and finding exact one-forms to be used as coordinate differentials we obtain

$$\begin{aligned} \xi^1 &= -\tfrac{1}{2}u^{-2}\,\mathrm{d}x + b\,\mathrm{d}t \\ \xi^2 &= -\mathrm{d}x \\ \xi^3 &= a\,\mathrm{d}t \\ \xi^4 &= \tfrac{1}{2}u^{-1}\,\mathrm{d}t \\ \xi^5 &= u\,\mathrm{d}t \end{aligned}$$

with functions u, a and b of the coordinates x and t ($g = 2$).

Plugging them into the ρ system of (35) yields the differential equations

$$\begin{aligned} u_t &= u^3 b_x \\ a_x &= b \\ u_x &= a \end{aligned}$$

and thus the Harry Dym equation $u_t = u^3 u_{xxx}$. On the other hand, there is another exact one-form contained in the ρ system:

$$\mathrm{d}z := u^{-1}\,\mathrm{d}x - (ub - \tfrac{1}{2}a^2)\,\mathrm{d}t.$$

Using z instead of x as coordinate (x becomes a potential) and setting $u =: \mathrm{e}^{\varphi}$ we get the *modified Korteweg–de Vries equation* in the form

$$\varphi_t = \varphi_{zzz} - \tfrac{1}{2}\varphi_z^3.$$

Thus we see that the well-known correspondence between the Harry Dym and the modified Korteweg–de Vries equation amounts to nothing but a different representation of the two-dimensional integral manifold of *one* CC ideal. In that sense, all coordinates are on an equal footing. In section 5, however, we shall see that only a certain pair of coordinates is invariant under a Bäcklund transformation.

4.2. *The affine algebra* $A_2^{(2)}$

We are considering the following one-forms being dual to generators of $A_2^{(2)}$ (cf (33)):

$$\xi^1 \text{ dual to } X_5^0 \qquad \xi^2 \text{ dual to } X_8^1 \qquad \xi^3 \text{ dual to } X_3^1$$
$$\xi^4 \text{ dual to } X_2^1 \qquad \xi^5 \text{ dual to } X_6^1 \qquad \xi^6 \text{ dual to } X_7^1$$
$$\xi^7 \text{ dual to } X_4^2.$$

The corresponding CC ideal reads

$$\begin{aligned}
&\mathrm{d}\xi^1 = 0 && \xi^1\xi^7 - 4\xi^2\xi^6 - \xi^3\xi^5 = 0\\
&\mathrm{d}\xi^2 = 0 && -2\xi^3\xi^6 + 3\xi^4\xi^5 = 0\\
&\mathrm{d}\xi^3 = -2\xi^1\xi^2 && \xi^3\xi^7 = 0\\
&\mathrm{d}\xi^4 = \xi^1\xi^3 && \xi^4\xi^7 = 0\\
&\mathrm{d}\xi^5 = \xi^1\xi^4 && \xi^5\xi^7 = 0\\
&\mathrm{d}\xi^6 = \xi^1\xi^5 && \xi^6\xi^7 = 0\\
&\mathrm{d}\xi^7 = -2\xi^2\xi^5 - \xi^3\xi^4.
\end{aligned}$$

We solve the algebraic σ system in a similar manner to that used previously and find the one-forms:

$$\begin{aligned}
\xi^1 &= \mathrm{d}x\\
\xi^2 &= \tfrac{1}{3}u^{-3/2}\,\mathrm{d}x - \tfrac{1}{2}e\,\mathrm{d}t\\
\xi^3 &= c\,\mathrm{d}t\\
\xi^4 &= b\,\mathrm{d}t\\
\xi^5 &= a\,\mathrm{d}t\\
\xi^6 &= u\,\mathrm{d}t\\
\xi^7 &= \tfrac{4}{3}u^{-1/2}\,\mathrm{d}t.
\end{aligned}$$

Inserting them into the ρ system and putting everything together gives

$$u_t = u^{5/2}u_{xxxxx}$$

which is (1) for $k = 4, n = 2 + (2/k) = 5/2$ and has already been found in [3]. Again the ρ system admits another exact one-form

$$\mathrm{d}z := u^{-1/2}\,\mathrm{d}x - (\tfrac{1}{4}b^2 + \tfrac{1}{2}eu - \tfrac{1}{2}ac)\,\mathrm{d}t.$$

Using z to replace x as coordinate and setting $u =: e^{-2\varphi}$ it can be immediately concluded that

$$\varphi_t = \varphi_{zzzzz} - 5\varphi_{zzz}\varphi_{zz} - 5\varphi_{zzz}\varphi_z^2 - 5\varphi_{zz}^2\varphi_z + \varphi_z^5 \tag{36}$$

which is the integrated version of the pseudopotential equation of the *Sawada–Kotera equation* and *Kaup–Kuperschmidt equation* [9].

5. The Bäcklund transformation

In the last two decades the concept of *Bäcklund transformations* has been successfully used for generating new solutions of physically interesting nonlinear partial differential equations from known ones. The *sine-Gordon equation, nonlinear Schrödinger equation* and *Korteweg–de Vries equation* could be mentioned as prominent examples.

An interesting discovery in 1979 [10] was the N-fold Bäcklund transformation for the equation governing axially symmetric stationary gravitational fields. The possibility of deriving the Bäcklund transformation in a purely algebraic manner was the main point of that article. In the following years Neugebauer, Kramer and Meinel applied this 'dressing' method to *Einstein–Maxwell fields* in general relativity and the *AKNS system* [11, 12].

We shall show below that one can generalize this method to Bäcklund transformations of CC ideals.

Let $\mathcal{G}$ be a finite-dimensional Lie algebra

$$\mathcal{G} = \operatorname{span}\{X_i, i = 1, \ldots, m\}$$

and $\underline{X}_1, \ldots, \underline{X}_m$ a corresponding *faithful, tracefree* and *linear* representation. Let $L(\mathcal{G})$ be the loop algebra

$$L(\mathcal{G}) := \mathcal{G} \otimes \mathbb{R}(\lambda, \lambda^{-1}).$$

$\underline{X}^i_j(\lambda) := \lambda^i \underline{X}_j$ is then a representation of the loop algebra $L(\mathcal{G})$.

Now consider the one-forms

$$\xi^j_i \text{ dual to } X^i_j \qquad -l \leqslant i \leqslant k$$

for some integers $-l, k \in \mathbb{N}$. The integrability condition for the matrix-valued one-form

$$\Omega(\lambda) := -\mathrm{d}\Phi(\lambda) + \underline{X}^i_j(\lambda)\xi^j_i\Phi(\lambda) \tag{37}$$

is again the vanishing of the CC ideal on the integral manifold.

How can we formulate a Bäcklund transformation for this CC ideal? To this end we construct a matrix

$$\tilde{\Phi}(\lambda) := P(\lambda)\Phi(\lambda)$$

such that $\mathrm{d}\tilde{\Phi}(\lambda)\tilde{\Phi}^{-1}(\lambda)$ is a polynomial in λ of degree k and in λ^{-1} of degree l and only consists of the matrices $\underline{X}_1, \ldots, \underline{X}_m$. Hence

$$\underline{X}^i_j(\lambda)\tilde{\xi}^j_i := \mathrm{d}\tilde{\Phi}(\lambda)\tilde{\Phi}^{-1}(\lambda)$$

is the Bäcklund transformation for the CC ideal.

Now we can formulate the nN-fold Bäcklund transformation of a CC ideal dual to $\mathcal{G} = \mathrm{sl}(n, \mathbb{R})$ with the $m = n^2 - 1$ tracefree matrices $\underline{X}_1, \ldots, \underline{X}_m$ [13]:

Theorem 1. (*nN-fold Bäcklund transformation*) Let

$$P(\lambda) := \sum_{r=0}^{N} \lambda^r Q_r \in \mathbb{R}^{n,n}$$

be determined by

$$\text{(i)}\ \det Q_N = \text{constant} \neq 0$$

$$\text{(ii)}\ P(\lambda_i)\Phi(\lambda_i)v_i = 0$$

for some constant

$$\text{scalars } \lambda_i \neq \lambda_j \qquad i,j = 1,\dots,nN, i \neq j$$

$$\text{vectors } v_i \qquad i = 1,\dots,nN$$

under the assumption

$$\det \Phi(\lambda_i) \neq 0 \qquad i = 1,\dots,nN.$$

Then the nN-fold Bäcklund transformation reads

$$\underline{X}_j^i(\lambda)\tilde{\xi}_i^j = P(\lambda)\underline{X}_j^i(\lambda)P^{-1}(\lambda)\xi_i^j + \mathrm{d}P(\lambda)P^{-1}(\lambda) \qquad \forall\lambda \neq \lambda_i \tag{38}$$

by sorting with respect to the matrices $\underline{X}_j$ and powers of λ.

Since every finite-dimensional Lie algebra is a matrix algebra we can regard every finite-dimensional Lie algebra as a subalgebra of a suitable $\mathrm{sl}(n,\mathbb{R})$. To be sure that the previously mentioned Bäcklund transformation only acts within a certain subalgebra (or within a certain subalgebra of the infinite-dimensional Lie algebra, e.g. graded subalgebra) one has to make further restrictions on the matrix $P(\lambda)$ and the constants λ_i and v_i. Sometimes it is therefore necessary to allow for double zeros λ_i. One can easily show that theorem 1 also holds for double zeros λ_i.

Finally it has to be emphasized that (i) is only a technical condition simplifying the proof and (ii) is a linear algebraic system of equations for the coefficients of the matrices Q_r, which may be solved via Cramer's rule.

Proof 1. The following lemma helps us to simplify the calculations [14]:

Lemma 1. Let $A, B \in \mathbb{R}^{n,n}$ and $p \in \mathbb{R}^n$. Then

$$\begin{pmatrix} Ap = 0 \\ Bp = 0 \end{pmatrix} \Rightarrow A\hat{B} = 0$$

holds, where $\hat{B}$ is the adjoint of B, i.e. $B\hat{B} = \det B\underline{1}$.

To remove all superfluous quantities we investigate the one-form Ω (respectively $\tilde{\Omega}$) on the integral manifold $\Omega = 0$ (respectively $\tilde{\Omega} = 0$).

(37) therefore reads

$$\mathrm{d}\Phi(\lambda) = X(\lambda)\Phi(\lambda)$$

with

$$X(\lambda) := \underline{X}^i_j(\lambda)\xi^j_i .$$

$X(\lambda)$ is a matrix-valued one-form and polynomial of degree k in λ and degree l in λ^{-1}:

$$X(\lambda) \in: \mathcal{P}(k, l).$$

Let

$$\tilde{\Phi}(\lambda) := P(\lambda)\Phi(\lambda) := \sum_{r=0}^{N} \lambda^r Q_r \Phi(\lambda)$$

and

$$\tilde{\Phi}(\lambda_i)v_i = 0 \qquad i = 1, \ldots, nN.$$

Hence it follows

$$\det P(\lambda_i) = 0 \qquad i = 1, \ldots, nN$$

if $\det \Phi(\lambda_i) \neq 0$. From $\det P(\lambda)$ being a polynomial of degree nN in λ we conclude

$$\det P(\lambda) = \det Q_N \prod_{i=1}^{nN} (\lambda - \lambda_i)$$

which assuming $\det Q_N =$ constant implies

$$\mathrm{d}(\det P(\lambda)) = 0.$$

With

$$\mathrm{Tr}(\mathrm{d}\tilde{\Phi}\tilde{\Phi}^{-1}) = \mathrm{d}(\ln \det \tilde{\Phi}) \qquad \lambda \neq \lambda_i$$

we find

$$\mathrm{Tr}(\mathrm{d}\tilde{\Phi}\tilde{\Phi}^{-1}) = 0$$

because $0 = \mathrm{Tr}\, X(\lambda) = \mathrm{Tr}(\mathrm{d}\Phi\Phi^{-1}) = \mathrm{d}(\ln \det \Phi)$.

Introducing the tracefree matrix-valued one-form

$$\tilde{X}(\lambda) := \mathrm{d}\tilde{\Phi}\tilde{\Phi}^{-1} \qquad \text{for } \lambda \neq \lambda_i$$

we note that with

$$\mathrm{d}\tilde{\Phi}\tilde{\Phi}^{-1} = P\,\mathrm{d}\Phi\Phi^{-1}P^{-1} + \mathrm{d}PP^{-1} = PX(\lambda)P^{-1} + \mathrm{d}PP^{-1} \qquad (39)$$

for $\lambda \neq \lambda_i$ it immediately follows

$$\mathrm{d}\tilde{\Phi}\hat{\tilde{\Phi}} = (PX(\lambda)\hat{P} + \mathrm{d}P\hat{P})\det\Phi.$$

Since $P(\lambda)$ is a polynomial of degree N in λ we derive from the definition of the adjoint:

$$\hat{P} \in \mathcal{P}((n-1)N, 0)$$

and hence

$$\tilde{\Phi}\hat{\tilde{\Phi}} \in \mathcal{P}(nN + k, l).$$

Condition (ii) of theorem 1 gives

$$\mathrm{d}\Phi(\lambda_i)v_i = \Phi(\lambda_i)v_i = 0$$

because $v_i =$ constant, $i = 1, \ldots, nN$, i.e.

$$\mathrm{d}\tilde{\Phi}\hat{\tilde{\Phi}} \sim \prod_{i=1}^{nN}(\lambda - \lambda_i)$$

and that is why

$$\mathrm{d}\tilde{\Phi}\tilde{\Phi}^{-1} = \frac{\mathrm{d}\tilde{\Phi}\hat{\tilde{\Phi}}}{\det\tilde{\Phi}} = (\det Q_N \det\Phi)^{-1}\frac{\mathrm{d}\tilde{\Phi}\hat{\tilde{\Phi}}}{\prod_{i=1}^{nN}(\lambda - \lambda_i)}$$

is regular at the zeros $\lambda = \lambda_i$. Hence

$$\tilde{X}(\lambda) = \mathrm{d}\tilde{\Phi}\tilde{\Phi}^{-1} \in \mathcal{P}(k, l).$$

$\tilde{X}(\lambda)$ is therefore again a tracefree matrix-valued one-form and polynomial of degree k in λ and degree l in λ^{-1}, i.e. there are one-forms $\tilde{\xi}_i^j$ such that

$$\tilde{X}(\lambda) = \underline{X}_j^i\tilde{\xi}_i^j \qquad -l \leqslant i \leqslant k.$$

Thus (39) yields the formula for the Bäcklund transformation. QED

5.1. Application to $A_1^{(1)}$

The one-form pertinent to the CC ideal (35) reads in the linear representation

$$\underline{X}_1 = \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \qquad \underline{X}_2 = \frac{1}{2}\begin{pmatrix}0 & 0\\ -1 & 0\end{pmatrix} \qquad \underline{X}_3 = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}:$$

$$\Omega = -\mathrm{d}\Phi(\lambda) + (\lambda\underline{X}_2\xi^1 + \lambda\underline{X}_3\xi^2 + \lambda^2\underline{X}_1\xi^3 + \lambda^3\underline{X}_2\xi^4 + \lambda^3\underline{X}_3\xi^5)\Phi(\lambda). \qquad (40)$$

The crucial point is that the prolongation algebra is graded, i.e. the algebra only consists of generators

$$X_1 \otimes \lambda^{2i} \qquad X_2 \otimes \lambda^{2i-1} \qquad X_3 \otimes \lambda^{2i-1} \qquad i \geqslant 1.$$

This leads to the equivalent statements

$X(\lambda)$ has the form as in (40)

and

(o) $X(\lambda)$ is a polynomial of third degree in λ

(iii) $X(0) = 0$

(iv) $\Phi(-\lambda) = M\Phi(\lambda)C(\lambda)$

with

$$M = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

and a constant matrix C depending only on λ ((iv) is due to $X(-\lambda) = MX(\lambda)M$).

For the twiddled quantities $\tilde{\Phi}(\lambda) = P(\lambda)\Phi(\lambda)$ and $\tilde{X}(\lambda) = \mathrm{d}\tilde{\Phi}(\lambda)\tilde{\Phi}^{-1}(\lambda)$, (o) is satisfied by theorem 1 and (iii) (respectively (iv)) by

(iii)′ $P(0) = \text{constant}$

(iv)′ $P(-\lambda) = c_0 M P(\lambda) M \qquad c_0 = \text{constant}.$

Defining

$$P(\lambda) := Q_N\left(\sum_{r=0}^{N-1}\lambda^r Q_r + \lambda^N \underline{1}\right)$$

and choosing zeros and vectors

$$\lambda_{N+i} := -\lambda_i$$
$$v_{N+i} := C^{-1}(\lambda_i) M v_i \qquad i = 1, \ldots, N$$

(ii) gives

$$P(\lambda_i)\Phi(\lambda_i)v_i$$
$$P(-\lambda_i)M\Phi(\lambda_i)v_i \qquad i = 1, \ldots, N$$

and therefore

$$P(\lambda_{N+i})M\Phi(\lambda_i)v_i$$
$$P(-\lambda_{N+i})M(M\Phi(\lambda_i)v_i) \qquad i = 1, \ldots, N.$$

By means of lemma 1 we then find

$$P(-\lambda_i)M\hat{P}(\lambda_i) = 0 \qquad i = 1, \ldots, 2N$$

and since $P(-\lambda)M\hat{P}(\lambda) \in \mathcal{P}(2N, 0)$:

$$P(-\lambda) = RP(\lambda)M \tag{41}$$

for a matrix R, which is independent of λ.

This yields immediately

$$Q_N = (-1)^N RQ_N M.$$

To guarantee (iv)′ and (i) we have to choose

$$Q_N = \begin{pmatrix} e^{\psi} & 0 \\ 0 & e^{-\psi} \end{pmatrix}$$

which implies $R = (-1)^N M$. From (41) we conclude for Q_0:

$$Q_0 = \begin{pmatrix} q & 0 \\ 0 & c_1 q^{-1} \end{pmatrix} \qquad \text{for } N \text{ even}$$

$$Q_0 = \begin{pmatrix} 0 & q \\ c_1 q^{-1} & 0 \end{pmatrix} \qquad \text{for } N \text{ odd}$$

c_1 = constant, because $\det P(0) = \det Q_N \det Q_0 = \text{constant}$.

Finally

$$\psi = -\ln q + \text{constant}$$

solves the remaining equation (iii)′.

From the highest power λ^{N+3} in (38) we now derive

$$\tilde{\xi}^4 = e^{-2\psi}\xi^4$$
$$\tilde{\xi}^5 = e^{2\psi}\xi^5$$

from which follows

$$\tilde{\varphi} = \varphi - 2\ln q + \text{constant}$$
$$d\tilde{t} = dt.$$

Analogously, lower powers of λ in (38) determine $\tilde{\xi}^1, \tilde{\xi}^2$ and $\tilde{\xi}^3$. One can explicitly show that $d\tilde{z} = dz$. Hence it turns out that the coordinates of the modified

Korteweg–de Vries equation are invariant under the N-fold Bäcklund transformation whereas the coordinate x of the Harry Dym equation is not. Its transformation can be read from

$$d\tilde{x} = \tilde{u}\, dz + \tilde{u}(\tilde{u}\tilde{b} - \tfrac{1}{2}\tilde{a}^2)\, dt.$$

The concept of CC ideals circumvents the need of fixing coordinates for the purpose of getting Bäcklund transformations. On the contrary, later evaluation of the Bäcklund transformation of the CC ideal in terms of coordinates tells which coordinates are transformed and which are invariant—for lack of a better expression the latter may be called 'good' coordinates.

Setting

$$y_i := \frac{(\Phi(\lambda_i)v_i)^1}{(\Phi(\lambda_i)v_i)^2}$$

we finally get from (ii) the formula

$$q = -\frac{\begin{vmatrix} \lambda_1^N y_1 & \lambda_1 & \lambda_1^2 y_1 & \lambda_1^3 & \cdots & \lambda_1^{N-1} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda_N^N y_N & \lambda_N & \lambda_N^2 y_N & \lambda_N^3 & \cdots & \lambda_N^{N-1} \end{vmatrix}}{\begin{vmatrix} y_1 & \lambda_1 & \lambda_1^2 y_1 & \lambda_1^3 & \cdots & \lambda_1^{N-1} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ y_N & \lambda_N & \lambda_N^2 y_N & \lambda_N^3 & \cdots & \lambda_N^{N-1} \end{vmatrix}} \qquad \text{for } N \text{ even}$$

$$q = -\frac{\begin{vmatrix} \lambda_1^N y_1 & \lambda_1 y_1 & \lambda_1^2 & \lambda_1^3 y_1 & \cdots & \lambda_1^{N-1} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda_N^N y_N & \lambda_N y_N & \lambda_N^2 & \lambda_N^3 y_N & \cdots & \lambda_N^{N-1} \end{vmatrix}}{\begin{vmatrix} 1 & \lambda_1 y_1 & \lambda_1^2 & \lambda_1^3 y_1 & \cdots & \lambda_1^{N-1} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_N y_N & \lambda_N^2 & \lambda_N^3 y_N & \cdots & \lambda_N^{N-1} \end{vmatrix}} \qquad \text{for } N \text{ odd.}$$

5.2. Application to $A_2^{(2)}$

A linear and tracefree representation of $\mathrm{sl}(3,\mathbb{R})$ is given by

$$\underline{X}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \underline{X}_5 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\underline{X}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \qquad \underline{X}_6 = \begin{pmatrix} 0 & 0 & -3 \\ 0 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix}$$

$$\underline{X}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix} \qquad \underline{X}_7 = \begin{pmatrix} 0 & 6 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\underline{X}_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -3 \\ -3 & 0 & 0 \end{pmatrix} \qquad \underline{X}_8 = \begin{pmatrix} 0 & 0 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

the corresponding one-form by

$$\Omega = -\mathrm{d}\Phi(\lambda)+(\underline{X}_5\xi^1+\lambda\underline{X}_8\xi^2+\lambda\underline{X}_3\xi^3+\lambda\underline{X}_2\xi^4+\lambda\underline{X}_6\xi^5+\lambda\underline{X}_7\xi^6+\lambda^2\underline{X}_4\xi^7)\Phi(\lambda). \tag{42}$$

The symmetry

$$X^T(\lambda) = -MX(-\lambda)M \tag{43}$$

is again due to the gradation of the prolongation algebra $A_2^{(2)}$, where

$$M = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

As a consequence $X(\lambda)$ can be defined by

(o) $X(\lambda)$ is a polynomial of second degree in λ

(iii) $X(\lambda)$ does *not* contain $\underline{X}_1^0, \underline{X}_4^0, \underline{X}_1^2, \underline{X}_5^2$

(iv) $\Phi^T(\lambda)M\Phi(-\lambda) = C(\lambda) \qquad C(\lambda) = \text{constant } (\lambda).$

Similarly as in 4.1 (iv) has to be satisfied by

$$(\text{iv})' \qquad P^T(\lambda)MP(-\lambda) = (\det P(\lambda)\det P(-\lambda))^{1/3}M.$$

Setting

$$P(\lambda) := Q_N\left(\sum_{r=0}^{N-1}\lambda^r Q_r + \lambda^N\right)$$
$$\lambda_{N+i} := -\lambda_i$$
$$\lambda_{2N+i} := -\lambda_i$$
$$w_i^1 := v_{N+i}$$
$$w_i^2 := v_{2N+i} \qquad i = 1, \ldots, N$$

with the restrictions

$$(\text{iv})'' \qquad Q_N^T M Q_N = M$$
$$(\text{iv})''' \qquad w_i^{1,2^T} C^T(\lambda_i)v_i = 0 \qquad i = 1, \ldots, N$$

again guarantees (iv)′.

In terms of pseudopotential vectors

$$y(\lambda_i) := \Phi(\lambda_i)v_i \qquad r(-\lambda_i) := \Phi(-\lambda_i)w_i^1 \qquad s(-\lambda_i) := \Phi(-\lambda_i)w_i^2$$

(iv)‴ reads

$$r^{\mathrm{T}}(-\lambda_i)My(\lambda_i)=0$$
$$s^{\mathrm{T}}(-\lambda_i)My(\lambda_i)=0 \qquad i=1,\ldots,N$$

which can be solved by (cf (43))

$$y(\lambda_i):=M[r(-\lambda_i)\times s(-\lambda_i)].$$

Without going into details (cf [13]) we state that

$$Q_N=\begin{pmatrix} \mathrm{e}^{\psi} & 0 & 0\\ \frac{1}{2}g^2\mathrm{e}^{\psi} & \mathrm{e}^{-\psi} & g\\ g\mathrm{e}^{\psi} & 0 & 1\end{pmatrix}$$
$$\mathrm{e}^{-\psi}=c_0(Q_0)_{11}$$
$$g=-c_0(Q_0)_{31} \qquad c_0=\text{constant}$$

satisfies (i) and (iii).

Evaluation of (38) gives again

$$\tilde{\xi}^6=\mathrm{e}^{2\psi}\xi^6$$
$$\tilde{\xi}^7=\mathrm{e}^{-\psi}\xi^7$$

and therefore

$$\tilde{\varphi}=\varphi-\psi$$
$$\mathrm{d}\tilde{t}=\mathrm{d}t.$$

Moreover $\mathrm{d}\tilde{z}=\mathrm{d}z$ as in 4.1. The remarks made at the end of 4.1 apply here as well.

Finally the linear algebraic system (ii) has

$$\psi=-\ln\left|\frac{\begin{vmatrix} -(\lambda_1)^N y_1^1 & y_1^2 & y_1^3 & \cdots & \lambda_1^{N-1}y_1^1 & \lambda_1^{N-1}y_1^2 & \lambda_1^{N-1}y_1^3\\ -(-\lambda_1)^N r_1^1 & r_1^2 & r_1^3 & \cdots & (-\lambda_1)^{N-1}r_1^1 & (-\lambda_1)^{N-1}r_1^2 & (-\lambda_1)^{N-1}r_1^3\\ -(-\lambda_1)^N s_1^1 & s_1^2 & s_1^3 & \cdots & (-\lambda_1)^{N-1}s_1^1 & (-\lambda_1)^{N-1}s_1^2 & (-\lambda_1)^{N-1}s_1^3\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ -(\lambda_N)^N y_N^1 & y_N^2 & y_N^3 & \cdots & \lambda_N^{N-1}y_N^1 & \lambda_N^{N-1}y_N^2 & \lambda_N^{N-1}y_N^3\\ -(-\lambda_N)^N r_N^1 & r_N^2 & r_N^3 & \cdots & (-\lambda_N)^{N-1}r_N^1 & (-\lambda_N)^{N-1}r_N^2 & (-\lambda_N)^{N-1}r_N^3\\ -(-\lambda_N)^N s_N^1 & s_N^2 & s_N^3 & \cdots & (-\lambda_N)^{N-1}s_N^1 & (-\lambda_N)^{N-1}s_N^2 & (-\lambda_N)^{N-1}s_N^3 \end{vmatrix}}{\begin{vmatrix} y_1^1 & y_1^2 & y_1^3 & \cdots & \lambda_1^{N-1}y_1^1 & \lambda_1^{N-1}y_1^2 & \lambda_1^{N-1}y_1^3\\ r_1^1 & r_1^2 & r_1^3 & \cdots & (-\lambda_1)^{N-1}r_1^1 & (-\lambda_1)^{N-1}r_1^2 & (-\lambda_1)^{N-1}r_1^3\\ s_1^1 & s_1^2 & s_1^3 & \cdots & (-\lambda_1)^{N-1}s_1^1 & (-\lambda_1)^{N-1}s_1^2 & (-\lambda_1)^{N-1}s_1^3\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ y_N^1 & y_N^2 & y_N^3 & \cdots & \lambda_N^{N-1}y_N^1 & \lambda_N^{N-1}y_N^2 & \lambda_N^{N-1}y_N^3\\ r_N^1 & r_N^2 & r_N^3 & \cdots & (-\lambda_N)^{N-1}r_N^1 & (-\lambda_N)^{N-1}r_N^2 & (-\lambda_N)^{N-1}r_N^3\\ s_N^1 & s_N^2 & s_N^3 & \cdots & (-\lambda_N)^{N-1}s_N^1 & (-\lambda_N)^{N-1}s_N^2 & (-\lambda_N)^{N-1}s_N^3 \end{vmatrix}}\right|$$

as solution.

6. Remarks

In the following a generalization of the methods given in the previous sections shall be mentioned. As already known the differential equations for axially symmetric stationary fields of general relativity (essentially the Ernst equation) can be written as a CC ideal. Application of the Wahlquist–Estabrook technique [1, 2] does not only lead to a loop algebra as prolongation algebra but the well-known semidirect sum of $A_1^{(1)}$ and the *Virasoro algebra*. Moreover the N-fold Bäcklund transformation of Neugebauer *et al* [10] admits a translation into the language of Bäcklund transformations of CC ideals.

How has a Bäcklund transformation of CC ideals to be formulated if we are dealing with the semidirect sum of a loop algebra $L(\mathcal{G}) = \mathcal{G} \otimes \mathbb{R}(\lambda, \lambda^{-1})$ and the Virasoro algebra?

To this end let us consider the commutator relations

$$[X_i^n, X_j^m] = [X_i, X_j]^{n+m}$$
$$[D^n, X_i^m] = mX_i^{n+m}$$
$$[D^n, D^m] = (m-n)D^{n+m}$$

with $X_i \in \mathcal{G}$.

Now we define the one-form

$$\Omega(\lambda) = -\mathrm{d}\Phi(\lambda) - \lambda^{n+1}\eta_n\Phi_\lambda(\lambda) + \lambda^m \underline{X}_i \xi_m^i \Phi(\lambda)$$

where Φ depends on λ as before, whereas the one-forms η_n and ξ_m^i do not.

It is easy to verify that the integrability condition $\mathrm{d}\Omega = 0 \bmod (\Omega, \Omega_\lambda)$ results in the vanishing of the CC ideal pertinent to the commutator relations above if we identify

ξ_m^i dual to X_i^m

η_n dual to D^n.

Then in order to get the desired Bäcklund transformation one has to make the most general Ansatz

$$\tilde{\Phi}(\lambda) = f(\lambda)P(\lambda)\Phi(\lambda) = f(\lambda)\sum_{r=0}^{N} \lambda^r Q_r \Phi(\lambda)$$

where f is a suitable function of λ alone.

We present no further analysis here since applications, e.g. to Ernst's equation, will be published elsewhere.

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